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DEPARTMENT OF MATHEMATICS

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Order of an element of a group: Let $(\boldsymbol{G},\cdot)$ be a group and a be any element of $\boldsymbol{G}$.

Then the order of the element a is defined as the least positive integer n such that $a^n = e$.

If there exists no positive integer n such that $a^n = e$, then we say that a is of infinite order or zero order.

We denote the order of a by $\boldsymbol{O(a)}$ or $|a|$.

Example:

- If $G = \{1, -1\}$ then G is a finite group under usual multiplication. Here $\boldsymbol{O(1) = 1}$ & $\boldsymbol{O(-1) = 2}$ since $(-1)^2 = 1$.
- If $G = \{1, \omega, \omega^2\}$ then G is a finite group under usual multiplication. Here $\boldsymbol{O(1) = 1}$ & $\boldsymbol{O(\omega) = 3}$ $(\because (\omega)^3 = 1$ and $\boldsymbol{O(\omega^2) = 1}(\because (\omega^2)^3 = 1)$

POINTS TO REMEMBER:

- The order of every element of a finite group is finite and is less than or equal to the order of the group.
- In a group $\boldsymbol{G}$, if $a \in \boldsymbol{G}$, then $\boldsymbol{O(a) = O(a^{-1})}$.
- If a is an element of a group $\boldsymbol{G}$ such that $\boldsymbol{O(a) = n}$ then

$$a^m = e \; iff \; n/m$$

Problem : The order of every element of a finite group is finite and is less than or equal to the order of the group.

Proof : Let $(G,.)$ be a finite group ,let $a \in \boldsymbol{G},$ by closure law we have $a^2, a^3, a^4, \ldots \ldots . \in G$ since G is finite ,all the positive integral powers of a.

Let $a^r = a^s$ where r,s $\in N$and r>s

$\therefore \quad a^r = a^s \Rightarrow a^{r-s} = a^0 = e \Rightarrow a^m = e$ where r-s=m

Since $r > s, m$ is a positive integers

$\therefore \exists$ a positive integers m such that $a^m = e$

Hence if n is the positive integral value of m such that $a^n = e$,then $O(a) = n$

$\therefore O(a)$ is finite .

Now to prove that $O(a) \leq O(G)$

If possible let $O(a) > O(G)$.

Let $O(a) = n$ by closure we have $a^2, a^3, a^4, \dots \dots . \in G$

No those two elements are equal ,for if possible Let $a^r = a^s$

then $a^{r-s} = e$ since O< $r - s < n$, $O(a) < n$ which is contradiction

Hence the n elements $a^2, a^3, a^4, \dots \dots . \in G$ are distinct elements of G

Hence $O(a) \leq O(G)$

Theorem: The order of any positive integral power of an element a in a group G cannot exceed the order of a i.e in a group G,$O(\mathrm{a}^{\mathrm{m}}) \leq \mathrm{O}(\mathrm{a}), a \in G$ and $m \in$ N.

Proof :Let $O(a) = n$. if m is a positive integer ,then $a^m \in G.O(a) = n, a^n = e \Rightarrow (a^n)^m = e^m \Rightarrow a^{mn} = e \Rightarrow (a^m)^n = e$

$\therefore O(a^m) \leq n \Rightarrow O(a^m) \leq O(a)$

Exercise Problems:

- For any two elements a,b € G where G is a group then $O(a) = O(b^{-1}ab)$
- Show that all groups of order 4 and less are commutative.
- If $a$is an element of a group G such that $O(a) = n$, then $a^m = e$ iff n/m

Thank you